SELF-SIMILAR SOLUTIONS FOR PLASMA DYNAMICS IN A HIGH-DENSITY PINCH

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Considerable interest has been expressed in recent times to undertake a study of powerful discharges, i.e., linear z pinches, in a high-density plasma. This interest is generated by the problem of developing high-intensity radiation sources and the utilization of z pinches in frozen-in deuterium fibers as a new and promising method to generate thermonuclear energy [1-4].

In order better to understand the physical processes taking place in the plasma of a z pinch, we must know the distribution of current, temperature, and density, all of these dependent on the dynamics of the plasma. Self-similar solutions describing the dynamics of the z pinch in approximation of nondissipative magnetohydrodynamics have been studied in [5-8]. Such an approximation is to be found in a relatively large current and in the low density of the plasma in a z pinch, when the velocity of motion and the temperature of the plasma are sufficiently large and the dissipative terms in the equations can be neglected to the extent that the magnetic Reynolds number is large ($R_m = uR/\nu_m \gg 1$).

In the present paper we examine the self-similar solutions for the dynamics of a z pinch in a dense plasma in the case of relatively small current and low plasma temperature for the pinch. The passage of the electric current in the z pinch, with the current of rather great density, leads to the heating and disintegration of the plasma, and under the conditions of the experiments from [1-4] the velocity of the dispersion and the plasma temperature are not great, so that $R_m \approx 1.2\cdot 10^{-7} R^2\tau^{-1}T_{eV}^{3/2} \lesssim 1$. The dynamics of the plasma for the z pinch in this case is determined from the Joule heat, and in the MHD equations we must take into consideration terms with finite conductivity. At the same time, over a broad range of changes in the parameters the viscosity of the plasma is insignificant and we can neglect the corresponding terms in the equations of motion and heat transfer for the case in which $Re = uR/\nu \gg 1$.

1. Initial Equations. Self-Similar Variables. For a dense plasma we will use the equations of magnetohydrodynamics in the single-temperature approximation ($T_i = T_e = T$) [9]:

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{u}) = 0; \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}\left\{[\mathbf{u}\times\mathbf{B}] - \frac{c\mathbf{R}}{en}\right\}; \tag{2}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{c\rho}[\mathbf{j}\times\mathbf{B}] - \frac{1}{\rho}\nabla P; \tag{3}$$

$$\frac{3}{2}n\left[\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T\right] + P\operatorname{div}\mathbf{u} = -\operatorname{div}\mathbf{q} + \frac{\mathbf{j}^2}{\sigma} + \frac{\mathbf{j}\cdot\mathbf{R}_T}{en}. \tag{4}$$

Here $\rho = m_i n$ is the density of the plasma; n is the density of the number of particles; $\mathbf{u}$ is the velocity; $P = 2nT$ is the pressure; $\mathbf{B}$ is the magnetic field; $\mathbf{q}$ is the density of the heat flow; $\mathbf{j} = (c/4\pi)\operatorname{rot}\mathbf{B}$ is the density of the electric current; $\mathbf{R}$ and $\mathbf{R}_T$ denote the force of friction and of heat; $\sigma = (e^2n/m_e)\tau_e$ is the conductivity of the plasma.

Let us examine the case of a nonmagnetized plasma, corresponding to the conditions of the experiment conducted in [1-4]. Using the explicit form of the kinetic coefficients from [9] and comparing the dissipative terms in the right-hand side of Eqs. (2)-(4) to each other, we can demonstrate that all of the dissipative terms are small in comparison with the Joule dissipation in proportion to the smallness of the parameters:

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$$\omega_e\tau_e \ll 1, \ \beta\omega_e\tau_e \ll 1 \quad (\beta = 4\pi P/B^2 \sim 1). \tag{5}$$

Neglecting the inertial terms in (3) for subsonic flows and, in view of (5), retaining in these equations only those terms with finite conductivity, we can rewrite (1)-(4) in cylindrical coordinates:

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(nru) = 0; \tag{6}$$

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial r}(uB) = \frac{\partial}{\partial r}\left(\frac{\nu_m}{r}\frac{\partial}{\partial r}(rB)\right); \tag{7}$$

$$\frac{1}{4\pi r}B\frac{\partial}{\partial r}(rB) + \frac{\partial P}{\partial r} = 0; \tag{8}$$

$$3n\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r}\right) + P\frac{1}{r}\frac{\partial}{\partial r}(ru) = \frac{\nu_m}{4\pi r^2}\left(\frac{\partial}{\partial r}(rB)\right)^2, \tag{9}$$

where $u \equiv u_r$ and $B \equiv B_\varphi$ denote the radial component of velocity and the azimuthal magnetic field; $\nu_m = c^2/(4\pi\sigma)$ is the magnetic viscosity.

The velocity of the plasma particles and the magnetic field vanish at the axis, and where $r = R(t)$ at the pinch boundary the density vanishes, while the magnetic field corresponds to a total current. The boundary conditions for system of equations (6)-(9) are thus as follows: $u = 0$, $B = 0$ with $r = 0$, $n = 0$; $u = \dot{R}(t)$, $B = (2I(t))/(cR(t))$ with $r = R(t)$. We will introduce the self-similar coordinate $\xi = r/R(t)$ and write solution (6)-(9) in the form

$$u(r, t) = \dot{R}(t)\xi, \ n(r, t) = n_0\alpha^{-2}(t)n_1(\xi),$$
$$T(r, t) = T_0\alpha^{-2\varkappa}(t)T_1(\xi), \ B(r, t) = B_0\alpha^{\mu}(t)B_1(\xi) \tag{10}$$

[$\alpha(t) = R(t)/R_0$, $R_0 = R(0)$ is the initial radius of the pinch, and μ and κ are indices of self-similarity].

Connecting the arbitrary dimensional constants by the relationship $16\pi n_0T_0 = B_0^2$ and making the transition in (6)-(9) to the self-similar variables, we find that (6) is satisfied identically, while the separation of the variables ξ and t in (8) requires satisfaction of the condition

$$\mu + \varkappa + 1 = 0. \tag{11}$$

With consideration of (11), we find that Eq. (8) in self-similar variables assumes the form

$$\frac{d}{d\xi}(n_1T_1 + B_1^2) + \frac{2}{\xi}B_1^2 = 0. \tag{12}$$

Making the transition to the self-similar variables and separating the variables in (9), we obtain

$$\left(1 - \frac{3}{2}\varkappa\right)\dot{\alpha} = \lambda^2\alpha^{3\varkappa-1}; \tag{13}$$

$$\left[\frac{1}{\xi}\frac{d}{d\xi}(\xi B_1)\right]^2 = \lambda^2 n_1 T_1^{5/2}, \tag{14}$$

where λ^2 is the separation constant; the dot denotes differentiation with respect to time, measured in units of $t_0 = R_0/\sqrt{\nu_m^{(0)}}$ ($\nu_m^{(0)} = \nu_m(t = 0, r = R_0)$).

The separation of variables in the induction equation (7) leads to the equations

$$(\mu + 1)\dot{\alpha} = 2\beta^2(\zeta + 1)\alpha^{3\varkappa-1}; \tag{15}$$

$$\frac{d}{d\xi}\left(T_1^{-3/2}\frac{1}{\xi}\frac{d}{d\xi}(\xi B_1)\right) = 2\beta^2(\zeta + 1)B_1 \tag{16}$$

(β is the separation constant, $\zeta = \lambda^2/\beta^2$).

The common solution of (13) and (15) imposes the condition $\mu = -1 - \kappa = -(4\zeta + 5)/(2\zeta + 3)$ on the constants μ and κ and leads to a time dependence of the functions in the self-similar solution

$$\alpha(t) = \left(1 + 2\lambda^2\frac{t}{t_0}\right)^{-1-3/2\zeta}. \tag{17}$$

The profiles of the self-similar variables $n_1(\xi)$, $T_1(\xi)$, $B_1(\xi)$ can be found from the solution of the system of ordinary differential equations (12), (14), (16) with boundary conditions $T_1(0) = n_1(0) = 1$, $B_1(0) = 0$.

Before we undertake the study of these solutions, let us take note of the fact that the possible self-similar solutions may represent the asymptote of the true motion of the plasma, provided that the time dependence of the current of the z pinch for such a solution is in agreement with the change in the current within the circuit, with an external current source for the z pinch.

Taking into consideration the self-similar representation for the density of the electric current, and namely:

$$j(r, t) = \frac{cB_0}{4\pi R_0}\alpha^{\mu-1}\frac{1}{\xi}\frac{d}{d\xi}(\xi B_1),$$

we find the expression for the time dependence of the total current in the z pinch:

$$I(t) = 2\pi\int_0^{R(t)} j(r, t)\, r\, dr = \frac{1}{2}cR(t)\, B(t)\lim_{\xi\to 1}\xi B_1(\xi). \tag{18}$$

According to (17) and (10), it follows from (18) that when $-1 < \zeta < 0$ the self-similar solutions correspond to the disintegration of the plasma as the temperature is reduced and as the density reaches the stage of total-current diminution, while when $-3/2 < \zeta < -1$ the self-similar solutions will correspond to an increase in the total current and to the disintegration of the plasma as density is reduced and temperature is raised. The solution corresponding to $\zeta > 0$ and $\zeta < -3/2$ may describe the self-similar compression regime as the total current, the temperature, and the density are increased.

2. Self-Similar Solutions. For a qualitative study of the solutions of Eqs. (12), (14), (16) it is convenient to exclude the density $n_1(\xi)$. In the equations for B_1 and T_1, derived after elimination of n_1, if we turn to the new variable $\eta = \xi^2$ and the new functions $Y = T_1^{-3/2}(\eta)$, $\psi = \eta^{1/2}B_1(\eta)$, we will have

$$\frac{d}{d\eta}\left(Y\frac{d\psi}{d\eta}\right) - \frac{\lambda^2}{2\zeta}(\zeta+1)\frac{1}{\eta}\psi = 0; \tag{19}$$

$$\frac{d}{d\eta}\left[Y\left(\frac{d\psi}{d\eta}\right)^2\right] + \frac{\lambda^2}{2}\frac{\psi}{\eta}\frac{d\psi}{d\eta} = 0. \tag{20}$$

The boundary conditions for Y and ψ are

$$\psi(0) = 0, \quad Y(0) = 1, \quad \frac{d\psi}{d\eta}(0) = \lambda/2, \tag{21}$$

where, without limiting generality, we can assume that $\lambda > 0$. With consideration of (21), we will find the first integral of system (19), (20):

$$\left(\frac{2}{\lambda}\frac{d\psi}{d\eta}\right)^{3\zeta+2} = Y^{-(2\zeta+1)}, \tag{22}$$

in particular,

$$Y = \text{const} = 1 \text{ when } \zeta = -2/3. \tag{23}$$

With the aid of (22), eliminating Y from (19), we will have

$$\frac{d}{d\eta}\left[\left(\frac{2}{\lambda}\frac{d\psi}{d\eta}\right)^{-\frac{3\zeta+2}{2\zeta+1}}\frac{d\psi}{d\eta}\right] - \frac{\lambda^2}{2\zeta}(\zeta+1)\frac{1}{\eta}\psi = 0. \tag{24}$$

Equation (24) belongs to the class of equal degree. Its order can be reduced in standard fashion.

Having introduced the new variable $y = (\lambda/2)^{\zeta/(3\zeta+2)}\eta^{-(\zeta+1)/(3\zeta+2)}\psi(\eta)$ and the new function $p(y) = dy/(d\ln\eta)$, from (24) when $\zeta \neq -2/3$ and $\zeta \neq -1/2$ we obtain a first-order differential equation for p(y):

$$p\frac{dp}{dy}-\frac{\zeta}{3\zeta+2}p-\frac{(\zeta+1)(2\zeta+1)}{(3\zeta+2)^2}y+\frac{4\zeta+2}{\zeta}y\left(\frac{\zeta+1}{3\zeta+2}y+p\right)^{\frac{3\zeta+2}{2\zeta+1}}=0. \qquad (25)$$

Conventional methods may be used to undertake a qualitative study of the behavior of the solutions for Eq. (25). The behavior of the integral curves of Eq. (25), as well as the form of the self-similar profiles depend significantly on the parameter ζ. With $\zeta < -1$ and $\zeta > 0$ the solutions correspond to the compression or disintegration of the plasma, whose density does not vanish at finite distances from the axis, but the total mass and current diverge, i.e., such solutions have no direct physical meaning. For $0 > \zeta > -1$ the solutions correspond to the dynamics of a plasma for a pinch with a sharp boundary. The asymptotic behavior of the self-similar profiles near the boundary of the pinch as $\xi \to \xi_0$, where ξ_0 is the boundary of the pinch, can be written in the form

$$\psi \sim \psi_0 - \psi_1(\xi_0^2 - \xi^2)^{-\zeta/(\zeta+1)},\ n_1 \sim (\xi_0 - \xi)^{(3\zeta+4)/(3\zeta+3)},$$
$$T_1 \sim (\xi_0 - \xi)^{-(6\zeta+4)/(3\zeta+3)},\ j_1 \sim (\xi_0 - \xi)^{-(2\zeta+1)/(\zeta+1)}.$$

For $-1/2 < \zeta < 0$ the temperature and density of the electric current increase with approach to the boundary of the pinch $\xi \to \xi_0$. For $-2/3 < \zeta < -1/2$ the density of the current vanishes as $\xi \to \xi_0$; however, temperature increases with approach to the boundary. When $-1 < \zeta < -2/3$ both the temperature and the density of the electric current vanish as $\xi \to \xi_0$. Examples of self-similar profiles for $\zeta = -0.2$ and -0.6 can be found in Figs. 1 and 2, respectively.

3. Analytical Solutions. With certain values of ζ the solution of system of equations (19), (20) can be obtained in closed analytical form. For $\zeta = -2/3$ from (19) and (23) we find

$$Y = 1,\quad \frac{d^2\psi}{d\eta^2} + \frac{\lambda^2}{4\eta}\psi = 0. \qquad (26)$$

The solution of (26) with boundary conditions (21) is $\psi = \xi J_1(\lambda\xi)$. Here $J_1(x)$ is the Bessel function, while the condition that the current density at the pinch boundary vanish when $\xi = \xi_0 = 1$ determines $\lambda = j_{0,1} = 2.405$, where $j_{0,1}$ is the first zero of the zeroth-order Bessel function.

Thus, the self-similar solutions for the case in which $\zeta = -2/3$ have the form

$$\alpha(t) = R(t)/R_0 = (1 + 2\lambda^2 t/t_0)^{5/4},\ T(\xi, t) = T_0\alpha^{-4/5},$$
$$n(\xi, t) = n_0\alpha^{-2}[J_0(\lambda\xi)]^2,\ B(\xi, t) = B_0\alpha^{-7/5}J_1(\lambda\xi),$$
$$j(\xi, t) = \frac{\lambda c B_0}{4\pi R_0}\alpha^{-12/5}J_0(\lambda\xi),\quad I(t) = cB_0R_0J_1(\lambda)\alpha^{-2/5}(t).$$

This solution describes the disintegration of the isothermal pinch as the total current drops in the circuit: $I(t) \sim t^{-1/2}$ when $t/t_0 \gg 1$.

For $\zeta = -1$ we find from Eqs. (22) and (19) that $\psi = (1/2)\lambda\eta$, $Y = 1 - (1/2)\lambda^2\eta$. Hence

$$R(t) = R_0(1 + 2\lambda^2 t/t_0)^2,\ n(\xi, t) = n_0\alpha^{-2}(1 - (1/2)\lambda^2\xi^2)^{5/3},$$

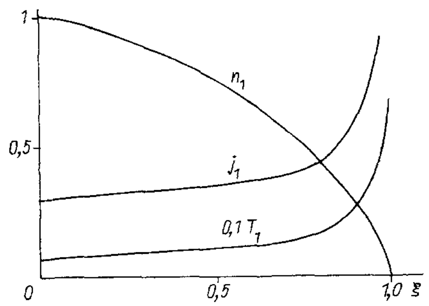

Fig. 1

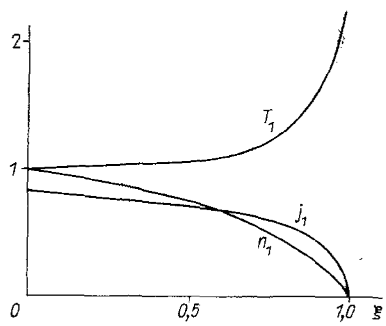

Fig. 2

$$T(\xi, t) = T_0\alpha^{-1}(1 - (1/2)\lambda^2\xi^2)^{-2/3},\ B(\xi, t) = B_0\alpha^{-2/3}(1/2)\lambda\xi,$$

$$j(\xi, t) = \frac{\lambda c B_0}{4\pi R_0}\alpha^{-5/2}(t),\quad I(t) = \frac{1}{4}c\lambda R_0 B_0\alpha^{-1/2}(t).$$

At the boundary of the pinch with $\xi = \xi_0 = 1$ the pressure and the density of the plasma vanish when $\lambda = \sqrt{2}$, which corresponds to the disintegration of the plasma filament of the pinch with a current density constant to the cross section, while the total current in the circuit drops and the plasma is overheated near the pinch boundary.

For $\zeta = -1$ the solution of Eqs. (22) and (19) has the form of $\psi = 2\lambda\eta/(4 + \lambda^2\eta)$, $Y = (1 + (1/4)\lambda^2\eta)^2$ and corresponds to a pinch with a diffused boundary, while the disintegration of the pinch occurs with a constant total current

$$\alpha(t) = (1 + 2\lambda^2 t/t_0)^{1/2},$$

$$n(\xi, t) = n_0\alpha^{-2}(1 + (1/4)\lambda^2\xi^2)^{-2/3},\ T(\xi, t) = T_0(1 + (1/4)\lambda^2\xi^2)^{-4/3},$$

$$B(\xi, t) = B_0\alpha^{-1}2\lambda\xi/(4 + \lambda^2\xi^2),\ I(t) = (c/\lambda)R_0B_0 = \text{const}.$$

In the latter case, the value of λ is determined by the magnitude of the total current flowing in the pinch.

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